AD-A063 970 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES B--ETC(U)
AUG 78 C T KELLEY DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1873 NL

1 OF 1
ADA
063970

END
DATE
FILMED
3-79
DDC

MRC-TSR-

MRC Technical Summary Report #1873

A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY SUMS OF FUNCTIONS OF ONE VARIABLE

C. T. Kelley

LEVEL II

DAAG29-75-C-0024, NSF-MCS75-17385

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

Aug[redacted]78

9p.

(Received July 3, 1978)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

221 200

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY SUMS OF FUNCTIONS OF ONE VARIABLE

C. T. Kelley

Technical Summary Report #1873
August 1978

ABSTRACT

For a class of functions of several variables, which contains the continuous functions, we show that there exists a sum of functions of one variable that minimizes the distance from the given function to the space of such sums. For functions of two variables we show that such a minimizing sum may be constructed by an iterative scheme.

AMS (MOS) Subject Classifications: 41A30, 47H10
Key Words: Compact map, Iteration, Minimizing sequence
Work Unit Number 2 (Other Mathematical Methods) or
6 (Spline Functions and Approximation Theory)

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS75-17385 A01.

SIGNIFICANCE AND EXPLANATION

Often it is desirable to approximate a given function as closely as possible by a member of a class of functions that are, in some sense, simpler than the original function.

In this paper we consider the approximation of a function of several variables by sums of functions of one variable relative to the supremum norm. It is not obvious that a best such approximation exists. We prove that such an approximation does, in fact, exist if the domain of the function to be approximated is a rectangle in a generalized sense, and if the function is in a certain class which includes the continuous functions. Also, if we consider functions of two variables, we show that a best such approximation may be found by an iterative method.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY SUMS OF FUNCTIONS OF ONE VARIABLE

C. T. Kelley

I. Introduction

Let $\{\Omega_i\}_{i=1}^m$ be compact subsets of the reals. Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$. Let $L_\infty(\Omega)$ be the Banach space of essentially bounded real-valued functions on Ω with the supremum norm. Let $S(\Omega)$ denote the closed subspace of $L_\infty(\Omega)$ consisting of sums of the form $\sum_{i=1}^m f_i$, with $f_i \in L_\infty(\Omega_i)$.

Let $K(\Omega)$ be the closure of all finite sums of the form $\sum_{k=1}^M \alpha_k \sum_{j=1}^m \varphi_{kj}$, where $\varphi_{kj} \in L_\infty(\Omega_j)$ for $1 \le j \le m$. $K(\Omega)$ is a closed subalgebra of $L_\infty(\Omega)$; $K(\Omega)$ contains the continuous functions on Ω; $S(\Omega) \subset K(\Omega)$. In fact, $K(\Omega)$ is the smallest subalgebra of $L_\infty(\Omega)$ containing $S(\Omega)$. For $k \in L_\infty(\Omega)$ we define a functional $\mu(k)$ by

$$\mu(k) = \inf_{f \in S(\Omega)} \|k - f\| . \tag{1.1}$$

In [1], Diliberto and Straus considered the problem of finding a sequence $\{f_n\} \subset S(\Omega)$ so that $\lim_{n\to\infty} \|k - f_n\| = \mu(k)$. They were able to do this and for continuous k, their sequence possessed a convergent subsequence. Hence, for continuous k, the infimum in (1.1) is attained. The purpose of this note is to show that the infimum in (1.1) is attained for all $k \in K(\Omega)$, and that for $m = 2$, the iteration scheme of Diliberto and Straus converges for all $k \in K(\Omega)$. These results partially answer questions raised in [1].

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS75-17385 A01.

II.

For $k \in L_\infty(\Omega)$ and $1 \leq j \leq m$ define $H_j(k) \in L_\infty(\Omega_j)$ by,

$$H_j(k)(x_j) = \frac{1}{2}\left(\operatorname*{ess\,sup}_{\substack{x_i \in \Omega_i \\ i \neq j}} k(x) + \operatorname*{ess\,inf}_{\substack{x_i \in \Omega_i \\ i \neq j}} k(x)\right), \text{ a.e. } x_j . \tag{2.1}$$

The sequence $\{f_n\}$ of Diliberto and Straus is defined as follows. Let $\{k_n\}$ be given by

(2.1)
$$
\begin{aligned}
k_0 &= k , \\
k_1 &= k - H_1(k) , \\
k_2 &= k - H_1(k) - H_2(k - H_1(k)) , \\
&\vdots \\
k_{mp+r} &= k_{mp+r-1} - H_r(k_{mp+r-1}) \quad \text{for } 1 \leq r \leq m .
\end{aligned}
$$

We define f_n by:

$$f_n = k - k_n . \tag{2.2}$$

The following theorem is due to Diliberto and Straus.

Theorem (2.1). For $k \in L_\infty(\Omega)$, let k_n be given by (2.1). Then

$$\lim_{n\to\infty} \|k_n\| = \lim_{n\to\infty} \|k - f_n\| = \mu(k) .$$

Moreover, for $n \geq 1$, $\|k_n\| \leq \|k_{n-1}\|$, and hence

$$\|f_n\| \leq 2\|k\| .$$

We list some obvious properties of the functions $H_i(k)$ in the following lemma.

Lemma (2.1). Let k $L_\infty(\Omega)$, let $\varphi_i \in L_\infty(\Omega_i)$, and let $\{E_r^i\}_{r=1}^R$ be finitely many disjoint measurable sets in Ω_i such that $\Omega_i = \bigcup_{r=1}^R E_r^i$. Let $\chi_{E_r^i}$ denote the characteristic function of E_r^i. Let $\{\alpha_{rs}\}_{s,r=1}^R$ be real, and let $\{p_s\}_{s=0}^R \subset L_\infty(\Omega)$ be independent of x_i. Then,

(a) $$H_i(k + \varphi_i) = H_i(k) + \varphi_i ,$$

(b) $$\| k - H_i(k) \| \le \|k\| ,$$

(c) $$H_i(\varphi_i,k) = \varphi_i H_i(k) ,$$

(d) $$H_i\left(\sum_{r=1}^{R} \sum_{s=1}^{R} \alpha_{rs}\chi_{E_r^i} p_s - p_0\right) = \sum_{r=1}^{R} \chi_{E_r^i} H_i\left(\sum_{s=1}^{R} \alpha_{rs} p_s - p_0\right) .$$

For $f \in S(\Omega)$, we may write $f = \sum_{i=1}^{m} \varphi_i$, with $\varphi_i \in L_\infty(\Omega_i)$. This representation of f is unique in the sense that if $f = \sum_{i=1}^{m} \varphi_i = \sum_{i=1}^{m} \psi_i$, then there are constants δ_i, so that $\sum_{i=1}^{m} \delta_i = 0$, and $\psi_i + \delta_i = \varphi_i$. Now let $f \in S(\Omega)$ and let $f = \sum_{i=1}^{m} \varphi_i$; $\varphi_i \in L_\infty(\Omega_i)$. For $k \in L_\infty(\Omega)$ define $Q_k(f) = \sum_{i=1}^{m} \psi_i$, where $\psi_i \in L_\infty(\Omega_i)$ is given by

(2.3) $$\psi_i = H_i(k - \sum_{j=1}^{i-1} \psi_j - \sum_{j=i+1}^{m} \varphi_j) .$$

Note that each individual ψ_i depends on the representation $f = \sum_{i=1}^{m} \varphi_i$. However, $Q_k(f)$ does not depend on the representation of f. Indeed if $f = \sum_{i=1}^{m} \varphi_i + \delta_i$, where the δ_i's are constants such that $\sum \delta_i = 0$, let $\hat{\psi}_i$ be defined by (2.3) with φ_i replaced by $\varphi_i + \delta_i$. We have $\hat{\psi}_1 = H_1(k - \sum_{i=2}^{m} (\varphi_i + \delta_i)) = H_1(k - \sum_{i=2}^{m} \varphi_i) = \sum_{i=2}^{m} \delta_i$ $= \psi_1 + \delta_1$. Hence $\hat{\psi}_2 = \psi_2 - \delta_1 - \sum_{i=3}^{m} \delta_i = \psi_2 + \delta_2$. Continuing in this way we obtain $\hat{\psi}_i = \psi_i + \delta_i$, for all i, and hence $\sum_{i=1}^{m} \hat{\psi}_i = \sum_{i=1}^{m} \psi_i$. Note that Q is continuous as a map on $S(\Omega)$.

For $k \in L_\infty(\Omega)$ let f_n be given by (2.2), then for $p \ge 1$

(2.4) $$f_{mp} = Q_k^p(0) .$$

Also by Theorem (2.1) for any $k \in L_\infty(\Omega)$ and $f \in S(\Omega)$ we have $\|k - Q_k(f)\| \le \|k - f\|$. Hence for each k, Q_k maps bounded sets in $S(\Omega)$ to bounded sets in $S(\Omega)$.

Theorem (2.2). Let $k \in K(\Omega)$. Q_k is a compact map on $S(\Omega)$. Hence $\{Q_k^p(0)\}_{p=1}^{\infty}$ has a convergent subsequence and the infimum in (1.1) is attained.

Proof. We give the proof for $m = 2$. The proof for arbitrary m is similar. Note first that for $j = 1,2$ and $k_1, k_2 \in L_\infty(\Omega)$, we have

$$\max_{x_j} |H_j(k_1)(x_j) - H_j(k_2)(x_j)| \leq \|k_1 - k_2\| . \tag{2.5}$$

If $f(x_1,x_2) = \varphi_1(x_1) + \varphi_2(x_2)$, then

$$Q_k(f) = H_1(k - \varphi_2) + H_2(k - H_1(k - \varphi_2)) . \tag{2.6}$$

Hence, for any $f \in S(\Omega)$, $k_1, k_2 \in L_\infty(\Omega)$,

$$\|Q_{k_1}(f) - Q_{k_2}(f)\| \leq 3\|k_1 - k_2\| . \tag{2.7}$$

Let $\varepsilon > 0$. As $k \in K(\Omega)$, we may find finitely many disjoint measurable sets $\{E_r^i\}_{r=1}^R$ in Ω_i such that $\bigcup_{r=1}^R E_r^i = \Omega_i$, and real numbers $\{\alpha_{rs}\}_{r,s=1}^R$, so that

$$\|k - \sum_{r,s=1}^R \alpha_{rs} \chi_{E_r^1} \chi_{E_s^2}\| < \varepsilon/3 . \tag{2.8}$$

Now let $\hat{k} = \sum_{r,s=1}^R \alpha_{rs}\chi_{E_r^1}\chi_{E_s^2}$. For $f = \varphi_1 + \varphi_2 \in S(\Omega)$, we have, by Lemma (2.1), that

$$Q_{\hat{k}}(f) = \sum_{r=1}^R \chi_{E_r^1} H_1\left(\sum_{s=1}^R \alpha_{rs}\chi_{E_s^2} - \varphi_2\right) + \sum_{s=1}^R \chi_{E_r^2} H_2\left(\sum_{r=1}^R \alpha_{rs}\chi_{E_r^1} - H_1(\hat{k} - \varphi_2)\right) .$$

As $\chi_{E_s^2}$ and φ_2 are independent of x_1, $H_1\left(\sum_{s=1}^R \alpha_{rs}\chi_{E_s^2} - \varphi_2\right)$ is constant for each r. Similarly $H_2\left(\sum_{r=1}^R \alpha_{rs}\chi_{E_r^1} - H_1(\hat{k} - \varphi_2)\right)$ is constant. Hence $Q_{\hat{k}}$ has finite dimensional range.

We apply (2.7) twice to obtain

$$\|Q_k(f) - Q_{\hat{k}}(f)\| \leq 3\|k - \hat{k}\| < \varepsilon . \tag{2.9}$$

Hence Q_k is the uniform limit of maps on $S(\Omega)$ which have finite dimensional range. This completes the proof.

We note that Theorem (2.2) is in a sense a converse of a theorem in [2]. Golomb showed, in a general Banach space setting, that if one assumes that the minimum in (1.1) is attained, then Theorem (2.1) holds.

The reader should note that if k is continuous on Ω, so is $Q^p(0)$ for each $p \geq 1$. Hence if k is continuous the infimum in (1.1) is attained at a continuous $f \in S(\Omega)$.

In order to prove the final result we require the following theorem of Diliberto and Straus.

Theorem (2.3). *Let* $m = 2$, $k \in K(\Omega)$. *Let* k_n *be given by* (2.1), *and let* k_* *be any limit point of the sequence* $\{k_n\}_{n=1}^{\infty}$. *We have*

$$H_1(k)(x_1) = 0, \quad H_2(k)(x_2) = 0 \quad \text{for a.e.} \quad (x_1,x_2) .$$

Theorem (2.4). *The sequence* $\{k_n\}$, *and hence the sequence* $\{f_n\}$, *converges in* $L_\infty(\Omega)$.

Proof. Let k_* be any limit point of the sequence $\{k_{2n}\}_{n=1}^{\infty}$. Write $k = k_* + \varphi_1 + \varphi_2$ with $\varphi_i \in L_\infty(\Omega_i)$. Then for $n \geq 2$ we have $k_{2n} = k_* + \varphi_1^{(n)} + \varphi_2^{(n)}$, and

$$\varphi_1^{(n)} = -H_1(k + \varphi_2^{(n-1)}) , \tag{2.10}$$

$$\varphi_2^{(n)} = -H_2(k + \varphi_1^{(n)}) .$$

As $H_1(k) = H_2(k) = 0$ a.e. we have, for $n \geq 2$

$$-\operatorname*{ess\,inf}_{x_2 \in \Omega_2} \varphi_2^{(n)}(x_2) \leq \operatorname*{ess\,sup}_{x_1 \in \Omega_1} \varphi_1^{(n)}(x_1) \leq -\operatorname*{ess\,inf}_{x_2 \in \Omega_2} \varphi_2^{(n-1)}(x_2) , \tag{2.11}$$

$$\operatorname*{ess\,sup}_{x_1 \in \Omega_1} \varphi_2^{(n)}(x_2) \leq -\operatorname*{ess\,inf}_{x_1 \in \Omega_1} \varphi_1^{(n)}(x_1) \leq \operatorname*{ess\,sup}_{x_2 \in \Omega_2} \varphi_2^{(n-1)}(x_2) .$$

Now there is, by assumption, a subsequence $\{k_{2n_j}\}_{j=1}^{\infty}$ of $\{k_{2n}\}_{n=1}^{\infty}$ which converges uniformly to k_*. This means $\lim_{j\to\infty} \|\varphi_1^{(n_j)} + \varphi_2^{(n_j)}\| = 0$. This in turn implies that there is a real number c, so that

$$\lim_{j\to\infty} \varphi_1^{(n_j)} = c = -\lim_{j\to\infty} \varphi_2^{(n_j)} . \tag{2.12}$$

Now choose $\varepsilon > 0$. There is j_0 so that $j \geq j_0$ implies that

$$\|\varphi_1^{(n_j)} - c\| < \varepsilon/2, \quad \|\varphi_2^{(n_j)} + c\| < \varepsilon/2 . \tag{2.13}$$

But (2.11) implies that, for all $k \geq 0$, and a.e. (x_1, x_2),

(2.14)
$$c - \varepsilon/2 \leq \varphi_1^{(n_{j0}+k)}(x_1) \leq c + \varepsilon/2 ,$$
$$-c - \varepsilon/2 \leq \varphi_2^{(n_{j0}+k)}(x_2) \leq -c + \varepsilon/2 .$$

Hence for all $n \geq n_{j0}$, we have

(2.15)
$$\|\varphi_1^{(n)} + \varphi_2^{(n)}\| \leq \varepsilon .$$

Hence $\lim_{n\to\infty} k_{2n} = k_*$. As $\lim_{n\to\infty} \|k_{2n} - k_{2n+1}\| = \lim_{n\to\infty} \|H_1(k_{2n})\| = 0$, $\lim_{n\to\infty} k_{2n+1} = k_*$. This completes the proof.

We note that these results generalize directly to the case where each Ω_i is a compact Hausdorff space endowed with a positive, regular Borel measure μ_i, and Ω is given the measure $\mu_1 \times \mu_2 \times \cdots \times \mu_n$. The functions k, $H_i(k)$, and f may be allowed to have values in $\mathbb{R}^n$ if supremums and infimums are understood to be taken componentwise.

Finally, the author would like to thank Professor Michael Golomb of MRC and Purdue University, Dr. Dennis Pence of MRC, and Professor M. G. Crandall of MRC and the University of Wisconsin for helpful discussions regarding this work.

REFERENCES

1) S. P. Diliberto and E. G. Straus, On the approximation of a function of several variables by a sum of functions of fewer variables. Pacific J. Math. 1 (1951), 195-210.

2) M. Golomb, Approximation by functions of fewer variables. "On Numerical Approximation", University of Wisconsin Press, Madison, WI, 1959, 275-327.

CTK/scr

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 1873	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): A NOTE ON THE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY SUMS OF FUNCTIONS OF ONE VARIABLE	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): C. T. Kelley	8. CONTRACT OR GRANT NUMBER(s): DAAG29-75-C-0024; MCS75-17385 A01
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit #'s 2 (Other Mathematical Methods) or 6 (Spline Functions & Approximation Theory)
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.	12. REPORT DATE: August 1978
	13. NUMBER OF PAGES: 6
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Compact map
Iteration
Minimizing sequence

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For a class of functions of several variables, which contains the continuous functions, we show that there exists a sum of functions of one variable that minimizes the distance from the given function to the space of such sums. For functions of two variables we show that such a minimizing sum may be constructed by an iterative scheme.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)